COMMUTATIVITY PATTERN OF FINITE NON-ABELIAN p-GROUPS DETERMINE THEIR ORDERS

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Abstract. Let G be a non-abelian group and $Z(G)$ be the center of G. Associate a graph Γ_G (called non-commuting graph of G) with G as follows: take $G\setminus Z(G)$ as the vertices of Γ_G and join two distinct vertices x and y, whenever $xy \neq yx$. Here, we prove that "the commutativity pattern of a finite non-abelian p-group determine its order among the class of groups"; this means that if P is a finite non-abelian p-group such that $\Gamma_P \cong \Gamma_H$ for some group H, then $|P| = |H|$.

1. Introduction and Results

Given a finite non-abelian group G, one can associate in many different ways a graph to G (e.g. [3, 11]). Here we consider the non-commuting graph Γ_G of G: the set of vertices of Γ_G is $G \setminus Z(G)$, and two vertices x and y are adjacent if and only if $xy \neq yx$. The non-commuting graph was first considered by Paul Erdös in 1975 [8]. The non-commuting graph of finite groups has been studied by many people (e.g., [1, 7]).
The non-commuting graph of a group is a discrete way to reflect the commutativity pattern of the group. In [1] the following conjecture was formulated:

Conjecture 1.1 (Conjecture 1.1 of [1]). *Let G and H be two finite non-abelian groups such that $\Gamma_G \cong \Gamma_H$. Then $|G| = |H|$.*

Conjecture 1.1 was refuted in [7] by exhibiting two groups G and H of orders

$$|G| = 2^{10} \cdot 5^3 \neq 2^3 \cdot 5^6 = |H|$$

with isomorphic non-commuting graphs.

In [1], it is proved that Conjecture 1.1 holds whenever one of the groups in question is a symmetric group, dihedral group, alternative group or a non-solvable AC-group (where by an AC-group we mean a group in which the centralizer of every non-central element is abelian). Recently Darafsheh [5] has proved the validity of Conjecture 1.1 whenever one of the groups G or H is a non-abelian finite simple group.

The main result of the present paper shows that any pair of groups consisting a counterexample for Conjecture 1.1 cannot contain a group of prime power order.

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Theorem 1.2. *If P is a finite non-abelian p-group such that $\Gamma_P \cong \Gamma_G$ for some group G, then $|P| = |G|$.*

This is a curious general phenomenon for non-abelian groups of prime power order: the order of a prime power order group can be determined among all finite groups by a proper model of its commutativity behavior, i.e, the non-commuting graph.

2. **Preliminary Results**

It is not hard to prove that the finiteness or the being non-abelian of a group can be transferred under graph isomorphism whenever two groups have the same non-commuting graph. Throughout P denotes a fixed but arbitrary finite non-abelian p-group of order p^n whose center $Z(P)$ is of order p^r and $1 < p^{a_1} < p^{a_2} < \cdots < p^{a_k}$ are all distinct conjugacy class sizes of P, where p^{a_i} is the size of conjugacy class ${g_i}^G$ of the element g_i. Throughout we also denote by u the greatest common divisor $\gcd(a_1, \ldots, a_k, n-r)$ of $\{a_1, \ldots, a_k, n-r\}$.

Lemma 2.1. *Let G be a finite non-abelian group and H be a group such that $\phi : \Gamma_G \to \Gamma_H$ is a graph isomorphism. Then the following hold:*

(1) $|C_H(h)|$ divides $(|g^G| - 1)(|Z(H)| - |Z(G)|)$, where $h = \phi(g)$.

(2) If $|Z(G)| \geq |Z(H)|$ and G contains a non-central element g such that $|C_G(g)|^2 \geq |G| \cdot |Z(G)|$, then $|G| = |H|$.

Proof. (1) Since $\Gamma_G \cong \Gamma_H$, we have

$$|G| - |Z(G)| = |H| - |Z(H)| \Rightarrow |H| = |G| - |Z(G)| + |Z(H)| \qquad (a)$$

and

$$|C_G(g)| - |Z(G)| = |C_H(h)| - |Z(H)| \Rightarrow |C_H(h)| = |C_G(g)| + |Z(H)| - |Z(G)| \quad (b)$$

As $|C_H(h)|$ divides $|H|$, $|C_H(h)|$ divides

$$(|C_G(g)| + |Z(H)| - |Z(G)|)\frac{|G|}{|C_G(g)|}, \qquad (c)$$

it follows from $(a), (b), (c)$ $|C_H(h)|$ divides

$$(|g^G| - 1)(|Z(H)| - |Z(G)|).$$

(2) Let $h = \phi(g)$. By part (1), we have $|C_H(h)| = |C_G(g)| + |Z(H)| - |Z(G)|$ divides $(|g^G| - 1)(|Z(H)| - |Z(G)|)$. Now, the inequality $|C_G(g)|^2 \geq |G||Z(G)|$ implies that

$$0 \leq |C_H(h)| \leq (|g^G| - 1)(|Z(G)| - |Z(H)|) < |C_G(g)| + |Z(H)| - |Z(G)| = |C_H(h)|$$

and this yields $(|g^G| - 1)(|Z(G)| - |Z(H)|) = 0$. Hence $|Z(G)| = |Z(H)|$. □

Lemma 2.2. *Suppose that $H = P_1 \times A$ is a finite group, where P_1 is a p-group, A is a finite abelian group such that* $\gcd(p, |A|) = 1$. *If $\Gamma_P \cong \Gamma_H$, then $|P| = |H|$.*

Proof. Let ϕ be a graph isomorphism from Γ_P to Γ_H. Suppose $h = \phi(g_t)$ for some $1 \leq t \leq k$ and $|P_1| = p^\kappa, |Z(P_1)| = p^\omega, |A| = a$ and $|C_H(h)| = ap^\nu$. Since $\Gamma_P \cong \Gamma_H$, we have

$$|P| - |Z(P)| = p^r(p^{n-r} - 1) = ap^\omega(p^{\kappa-\omega} - 1) = |H| - |Z(H)|,$$

$$|P| - |C_P(g_t)| = p^{n-a_t}(p^{a_t} - 1) = ap^\nu(p^{\kappa-\nu} - 1) = |H| - |C_H(h)|,$$

since $\gcd(a,p)=1$, it follows that $r=\omega$ and $n-a_t=\nu$. Therefore

$$|C_P(g_t)|-|Z(P)|=p^r(p^{n-a_t-r}-1)=ap^{\omega}(p^{\nu-\omega}-1)=|C_H(h)|-|Z(H)|.$$

Therefore $a=1$. Since $r=\omega$, $|Z(P)|=|Z(H)|$. Hence $|P|=|H|$. $\square$

Lemma 2.3. *Suppose $H=Q\times A$, where Q is a q-group for some prime q, A is an abelian group and* $\gcd(|A|,q)=1$. *If $\Gamma_P\cong\Gamma_H$, then $|H|=|P|$.*

Proof. If $p=q$, then Lemma 2.2 completes the proof. Suppose, for a contradiction, that $p\neq q$.

Note that $|g_1^P|=p^{a_1}$. Let ϕ be a graph isomorphism from Γ_P to Γ_H and let

$$\phi(g_1)=h, |A|=a, |Q|=q^{\kappa}, |C_H(h)|=aq^{\nu}, |Z(H)|=aq^{\omega}.$$

It is clear that $\kappa>\nu>\omega$. Since $\Gamma_P\cong\Gamma_H$, we have

$$|C_H(h)|-|Z(H)|=aq^{\omega}(q^{\nu-\omega}-1)=p^r(p^{n-a_1-r}-1)=|C_P(g_1)|-|Z(P)|, \tag{1}$$

$$|H|-|C_H(h)|=aq^{\nu}(q^{\kappa-\nu}-1)=p^{n-a_1}(p^{a_1}-1)=|P|-|C_P(g_1)|. \tag{2}$$

Since $|g_1^P|\leq|g^P|$ for all $g\in P\backslash Z(P)$, h^H has the minimum size among all conjugacy classes of non-central elements of H. By considering the conjugacy class equation of H, we have

$$aq^{\kappa}=aq^{\omega}+q^{\kappa-\nu}+\sum_{i=1}^{s}|{x_i}^H|,$$

where

$$\{{x_i}^H\mid i=1,\ldots,s\}=\{g^H\mid g\in H\setminus Z(H)\}\setminus\{h^H\}.$$

Since $\gcd(a,q)=1$ and $q^{\kappa-\nu}|\sum_{i=1}^{s}|{x_i}^H|$, it follows that

$$\kappa-\nu\leq\omega. \tag{$*$}$$

Equation (1) implies that the largest p-power number possibly dividing a is p^r. Now it follows from Equations (1), (2) and the inequality $(*)$ that

$$p^{n-a_1-r}|q^{\kappa-\nu}-1\leq q^{\omega}-1\leq p^{n-a_1-r}-2,$$

which is a contradiction. This completes the proof. $\square$

Lemma 2.4. *Let H be a group such that $\Gamma_P\cong\Gamma_H$. Then $|Z(H)|$ divides $p^r(p^u-1)$, where $u=\gcd(a_1,\ldots,a_k,n-r)$.*

Proof. (1) Since $\Gamma_P\cong\Gamma_H$, $|P|-|Z(P)|=|H|-|Z(H)|$ and $|P|-|C_P(g_i)|=|H|-|C_H(h_i)|$, for every $i\in\{1,\ldots,k\}$ and $h_i=\phi(g_1)$, where $\phi:\Gamma_G\to\Gamma_H$. Therefore we have the following equalities

$$p^r(p^{n-r}-1)=|Z(H)|\big(\frac{|H|}{|Z(H)|}-1\big)$$

$$p^{n-a_i}(p^{a_i}-1)=|C_H(h_i)|\big(\frac{|H|}{|C_H(h_i|)}-1\big)$$

for each $i\in\{1,\ldots,k\}$. Thus $|Z(H)|$ divides the great common divisors of the left hand side of two latter equalities which is $p^r(p^u-1)$. $\square$

A class of groups arising in the proof of our main result is the class of AC-groups; as we mentioned, a group G is called an AC-group whenever the centralizer of every non-central element is abelian. AC-groups was studied by many people (e.g., [10]). It is easy to see that $C_G(x) \cap C_G(y) = Z(G)$ for any two non-central elements $x, y \in G$ with distinct centralizers. This implies that

$$\mathfrak{C}(G) = \{C_G(x)/Z(G) \mid x \in G \setminus Z(G)\}$$

is a *partition* of $G/Z(G)$; where by a partition for a group H we mean a collection $\mathcal{C}$ of proper subgroups of H such that $H = \bigcup_{S \in \mathcal{C}} S$ and $S \cap T = 1$ for any two distinct $S, T \in \mathcal{C}$. Each element of $\mathcal{C}$ is called a component of the partition. If each component is abelian, we call $\mathcal{C}$ an abelian partition. Thus $\mathfrak{C}(G)$ is an abelian partition for $G/Z(G)$. The size of $\mathfrak{C}(G)$ is an invariant of the non-commuting graph Γ_G, called the clique number; where by definition the clique number of a finite graph is the maximum number of vertices which are pairwise adjacent. The clique number of the non-commuting graph Γ_H of a non-abelian group H will be denoted by $\omega(H)$. Thus $\omega(H)$ is simply the maximum number of pairwise non-commuting elements in the group.

Lemma 2.5. *Suppose that G is a finite non-abelian AC-group such that $G/Z(G)$ is a p-group. Then $\omega(G) \equiv 1 \mod p$.*

Proof. Since G is an AC-group, $\omega = \omega(G) = |\mathfrak{C}(G)|$, where

$$\mathfrak{C}(G) = \{C_G(x) \mid x \in G \setminus Z(G)\}.$$

On the other hand, $C_G(x) \cap C_G(y) = Z(G)$ for any two non-central elements $x, y \in G$ such that $C_G(x) \neq C_G(y)$. Therefore

$$|G| = -(\omega - 1)|Z(G)| + \sum_{S \in \mathfrak{C}(G)} |S|.$$

This completes the proof. □

Lemma 2.6. *[Mann [2]: Lemma 39.8, p. 354] Suppose C is a subgroup of group G and let $a \in G$ be such that $CC^a = C^aC$. Then $CC^a = C[C, a]$.*

Proof. We have

$$CC^a = \bigcup_{c \in C} Cc^a = \bigcup_{c \in C} Cc^{-1}c^a = \bigcup_{c \in C} C[c, a] \subseteq C[C, a].$$

Thus $CC^a \subseteq C[C, a]$. Since all the generators $[c, a] = c^{-1}c^a$ $(c \in C)$ of $[C, a]$ belongs to CC^a, we have $C[C, a] \subseteq CC^a$, since by hypothesis CC^a is a group. This completes the proof. □

In the following proposition we will use this property of any AC-groups G; for any two commuting non-central elements x and y of G, we have $C_G(x) = C_G(y)$.

Proposition 2.7. *Let G be a nilpotent AC-group of nilpotency class greater than 2, then the set $\mathfrak{C}$ of all centralizers of non-central elements of G has exactly one normal member T in G. In particular, T is a characteristic subgroup of G. Moreover, the latter normal subgroup T has the maximum order among all members of $\mathfrak{C}$.*

Proof. Let x be any element of $Z_2(G) \setminus Z(G)$. Then $C_G(x)$ is a normal subgroup of G containing G': for the map ϕ defined on G by $g^\phi = [x, g]$ for all $g \in G$ is a group homomorphism and its image is contained in $Z(G)$ and its kernel is $C_G(x)$.

Since G is of nilpotency class greater than 2, there exists an element $g \in G' \setminus Z(G)$. Since $[Z_2(G), G'] = 1$, the remark preceding the proposition implies that

$$C_G(x) = C_G(g) \text{ for all } x \in Z_2(G) \setminus Z(G). \qquad \diamondsuit$$

Now suppose that $N = C_G(y)$ is a normal centralizer of G for some non-central element y. Then there exists an element $t \in \big(N \cap Z_2(G)\big) \setminus Z(G)$, since $Z(G) \subsetneqq N$. Since $yt = ty$, it follows from $\diamondsuit$ that $C_G(t) = C_G(y) = C_G(g)$. Hence, we have so far proved that $\mathfrak{C}$ has exactly one normal member in G. This implies that $C_G(x)$ is a characteristic subgroup of G.

Now, we prove $C_G(x)$ has the maximum order among all members of $\mathfrak{C}$. Suppose that $C = C_G(h)$ for some $h \in G \setminus Z(G)$. We may assume that C is not normal in G. Thus there exists an element $a \in N_G(N_G(C)) \setminus N_G(C)$, since G is nilpotent. Then $C^a \neq C$, and C^a is a subgroup of $N_G(C)$. Let $A = CC^a$. By Lemma 2.6, we have

$$CC_G(x) \supseteq CG'Z(G) \supseteq C[C,a]Z(G) = CC^aZ(G) = CC^a = A.$$

It follows that

$$\frac{|C||C_G(x)|}{|Z(G)|} = |CC_G(x)| \geq |A| = |CC^a| = \frac{|C|^2}{|Z(G)|}$$

Thus $|C_G(x)| \geq |C|$. This completes the proof. □

The proof of existence of unique normal centralizer is due to Rocke [9, Lemma 3.8]; the argument to prove the existence of a normal centralizer of maximal order is due to Mann [2, Theorem 39.7, p. 354]. He has proved among all abelian subgroups of maximal order in a metabelian p-group, there exists a normal subgroup. The latter was first proved by Gillam [4].

Lemma 2.8. *Let P be of nilpotency class 2. Then $a_i \leq r$ for every i.*

Proof. Since P is of nilpotency class 2, for every $x \in P \setminus Z(P)$ with class size p^{a_i}, the conjugacy class of x is contained in $xP' \subseteq xZ(P)$. Hence $p^{a_i} \leq p^r$. This completes the proof. □

Now we will need the following two well known results about Frobenius groups.

Proposition 2.9.
1. *(see e.g., Theorem 6.7 of [6]) Let N be a normal subgroup of a finite group G, and suppose that $C_G(n) \subseteq N$ for every non-identity element $n \in N$. Then N is complemented in G, and if $1 < N < G$, then G is a Frobenius group with kernel N.*
2. *(see e.g., Lemma 6.1 of [6]) Let H be a Frobenius group with the kernel F and a complement K, then $|K|$ divides $|F| - 1$.*

Lemma 2.10. *Let $H = KF$ be a Frobenius group with the kernel F and a complement K. Suppose $1 \subset F_1 \subseteq F$ is a normal subgroup of H. Then $H_1 = KF_1$ is a Frobenius group with the kernel F_1 and a complement K.*

Proof. For every non-identity element f_1 of F_1,

$$C_{H_1}(f_1) = C_H(f_1) \bigcap H_1 \subseteq F \bigcap H_1 = F \bigcap F_1K = F_1,$$

by the Dedekind modular law. Therefore H_1 is a Frobenius group with the kernel F_1. It is clear that K is a complement for F_1 in H_1. □

3. **Proof of the Main Result**

In this section we prove our main result, Theorem 1.2.

We argue by induction on the order of P. If $|P| = p^3$, then $|P| = |G|$ by Proposition 3.20 of [1]. If P is not an AC-group, there exists a non-central element $x \in P$ such that $C_P(x)$ is non-abelian. If $y = \phi(x)$, then $\Gamma_{C_P(x)} \cong \Gamma_{C_G(y)}$. Now induction hypothesis implies that $|C_P(x)| = |C_G(y)|$ and since $|P| - |C_P(x)| = |G| - |C_G(y)|$, we have $|P| = |G|$. Thus, we may assume that P is an AC-group and so G is also an AC-group. By Proposition 3.14 of [1], we may assume that G is solvable. Therefore by the classification of non-abelian solvable AC-groups in [10], G is isomorphic to one the following groups H_i $(i = 1, \ldots, 5)$:

(1) H_1 is non-nilpotent and it has an abelian normal subgroup N of prime index and $\omega(H_1) = |N : Z(H_1)| + 1$.
(2) $H_2/Z(H_2)$ is a Frobenius group with the Frobenius kernel and complement $F/Z(H_2)$ and $K/Z(H_2)$, respectively and F and K are abelian subgroups of H_2 and $\omega(H_2) = |F : Z(H_2)| + 1$.
(3) $H_3/Z(H_3) \cong S_4$ and V is a non-abelian subgroup of H_3 such that $V/Z(H_3)$ is the Klein 4-group of $H_3/Z(H_3)$ and $\omega(H_3) = 13$, where S_4 is the symmetric group of on 4 letters.
(4) $H_4 = A \times Q$, where A is an abelian subgroup and Q is an AC-group of prime power order.
(5) $H_5/Z(H_5)$ is a Frobenius group with the Frobenius kernel and complement $F/Z(H_5)$ and $K/Z(H_5)$, respectively and K is an abelian subgroup of H. $Z(F) = Z(H_5)$, and $F/Z(H_5)$ is of prime power order and $\omega(H_5) = |F : Z(H_5)| + \omega(F)$.

By Lemmas 3.11 and 3.12 of [1] and Lemma 2.3, we may assume that G is isomorphic to either H_1 or H_5. Suppose that $G \cong H_1$. Then, obviously $\Gamma_P \cong \Gamma_{H_1}$. Since N is abelian, there exists $h \in H_1 \setminus Z(H_1)$ such that $C_{H_1}(h) = N$. As P is an AC-p-group, it follows from Lemma 2.5 that

$$\omega(P) \equiv 1 \mod p.$$

Since $\Gamma_P \cong \Gamma_{H_1}$, we have

$$\omega(H_1) = |C_{H_1}(h) : Z(H_1)| + 1 \equiv 1 \mod p,$$

and so $p \mid |C_{H_1}(h) : Z(H_1)|$. On the other hand Lemma 2.1(1) implies that, $|C_{H_1}(h)|$ divides $(p^{a_t} - 1)(p^r - |Z(H_1)|)$, where g_t maps to h under a graph isomorphism from Γ_P to Γ_{H_1}. Thus p divides $|Z(H_1)|$ and so $p^2 \mid |C_{H_1}(h)|$. This follows that p^2 divides $|Z(H_1)|$. By continuing this latter process, one obtains that p^r divides $|Z(H_1)|$ and so $|Z(H_1)| \geq |Z(P)|$. Now, let $y \in H_1 \setminus C_{H_1}(h)$ so that $H_1 = C_{H_1}(h)C_{H_1}(y)$ and

$$|H_1||Z(H_1)| = |C_{H_1}(h)||C_{H_1}(y)| \leq \max\{|C_{H_1}(h)|^2, |C_{H_1}(y)|^2\}.$$

Now, Lemma 2.1(2) implies that $|P| = |H_1|$.

Thus, it remains to deal with the case $G \cong H_5$. Let $H = H_5$ and note that $\Gamma_P \cong \Gamma_H$. We need to introduce some new notation for the group H. Since $F/Z(F)$ is a q-group for some prime q, we set $|F| = bq^\kappa$, for some positive integer b such that $\gcd(b, q) = 1$ and therefore $|Z(H)| = bq^\omega$ and $|C_F(f_i)| = |C_H(f_i)| = bq^{\nu_i}$ for some $f_i \in F \setminus Z(F)$.(Recall that $Z(F) = Z(H)$ in this case) Since F is nilpotent and non-abelian, we have $1 \leq \omega < \nu_i < \kappa$. Since $\gcd(|K/Z(H)|, |F/Z(H)|) = 1$, we have $|C_H(h)| = |K| = aq^\omega$ for some $h \in H \setminus F$ and for some positive integer a.

It is clear that $b \mid a$ and $\gcd(a,q)=1$. Therefore $|H| = aq^{\kappa}$. Suppose that under a graph isomorphism from Γ_H to Γ_P, h maps to g_t for some integer $1 \le t \le k$ and f_i maps to g_i, where $1 \le i \le k$ and $i \neq t$. Here note that f_t is not defined. Suppose further that $\beta = a_t$.

We need to prove the following (a), (b), (c) and (d).

(a) $p \neq q$.

(b) if p^l divides a, for some integer l, then p^l divides b and p^{r+1} does not divide a. This simply means that the largest p-power part of a and b are the same and p^r is the largest p-power possibly dividing a.

(c) Γ_F is a regular graph so that there exists integers ν and α such that $\nu_i = \nu$ and $a_i = \alpha$ for all $1 \le i \le k$ and $i \neq t$.

(d) $\nu \le 2\omega$ and $\kappa \le 3\omega$.

Proof of (a) Suppose $p = q$. Since $\Gamma_P \cong \Gamma_H$, $ap^{\omega}(p^{\kappa-\omega}-1) = p^{n-\beta}(p^{\beta}-1)$ and $bp^{\omega}(p^{\nu_i-\omega}-1) = p^r(p^{n-a_i-r}-1)$. Therefore $n-\beta = \omega = r$, a contradiction.

Proof of (b) Since $\Gamma_P \cong \Gamma_H$, we have

$$(a-b)q^{\omega} = p^r(p^{n-\beta-r}-1). \tag{3}$$

Thus $p^r \mid a-b$. This proves the first part of **(b)** for all $l \in \{1,\ldots,r\}$. Now, suppose $t > r$ and p^t divides a and $p^t \nmid b$. Equation (3) shows $p^{r+1} \nmid b$. Now let $i \in \{1,\ldots,k\}$ such that $i \neq t$. Then by the graph isomorphism, we have

$$p^{n-a_t} - p^{n-a_i} = aq^{\omega} - bq^{\nu_i}. \tag{$**$}$$

Since $r+1 \ge n-a_t$ and $r+1 \ge n-a_i$, it follows from $(**)$ that p^{r+1} divides b, a contradiction. Now Equation (3) implies that $p^{r+1} \nmid a$ and since b divides a, the proof of part (b) follows.

Proof of (c) Suppose Γ_F is not regular. Therefore F has two centralizers $C_H(f_{i_1})$ and $C_H(f_{i_2})$ of order $bq^{\nu_{i_1}}$ and $bq^{\nu_{i_2}}$, respectively, where $\nu_{i_1} \neq \nu_{i_2}$. We may assume that the conjugacy class of f_{i_1} in F is of minimum size among all conjugacy classes of non-central elements of F. We distinguish two cases to reach a contradiction.

(I) Suppose that $\nu_{i_1} - \nu_{i_2} \le \omega$.

$$p^{n-a_{i_2}} - p^r = bq^{\nu_{i_2}} - bq^{\omega} \tag{4}$$

$$p^{n-a_{i_1}} - p^{n-a_{i_2}} = bq^{\nu_{i_1}} - bq^{\nu_{i_2}} \tag{5}$$

Now it follows from Equations (4),(5) and part **(b)** that

$$p^{n-a_{i_2}-r} | q^{\nu_{i_1}-\nu_{i_2}} - 1 \le q^{\omega} - 1 \le p^{n-a_{i_2}-r} - 2,$$

a contradiction.

(II) Suppose that $\nu_{i_1} - \nu_{i_2} > \omega$.

We claim that the nilpotency class of F is greater than 2. If not, then Lemma 2.8 implies that

$$\kappa - \nu_{i_2} \le \omega. \tag{$\clubsuit$}$$

Since $\nu_{i_1} - \nu_{i_2} > \omega$, $\clubsuit$ is a contradiction. Therefore the nilpotency class of F is greater than 2. Since H is an AC-group, F is also an AC-group. Therefore every maximal abelian subgroup of F is centralizer of non-central element of F.

By Proposition 2.7, F has a characteristic centralizer $C_F(f_j)$ of order $bq^{\nu_j} = bq^{\nu_{i_1}}$ having the maximum order among the proper centralizers. Thus $\nu_{i_1} = \nu_j$ and so $a_{i_1} = a_j$. Since F is normal subgroup of H, $C_F(f_j)$ is normal in H. Since $H/Z(H)$ is Frobenius group, by Lemma 2.10 $K/Z(H)C_F(f_j)/Z(H)$ is a Frobenius group with the kernel $C_F(f_j)/Z(H)$ and a complement $K/Z(H)$. Thus

$$\frac{a}{b}|q^{\nu_{i_1}-\omega} - 1. \qquad \heartsuit$$

By the graph isomorphism, we have

$$bq^{\omega}(q^{\nu_{i_1}-\omega} - 1) = p^r(p^{n-a_{i_1}-r} - 1), \tag{6}$$

$$bq^{\nu_{i_1}}(\frac{a}{b}q^{\kappa-\nu_{i_1}} - 1) = p^{n-a_{i_1}}(p^{a_{i_1}} - 1). \tag{7}$$

Since $\gcd(\frac{a}{b}, p) = 1$, Equations $\heartsuit$ and (6) imply that $\frac{a}{b}q^{\omega}|p^{n-a_{i_1}-r} - 1$. Equation (7) imply that $p^{n-a_{i_1}-r}|\frac{a}{b}q^{\kappa-\nu_{i_1}} - 1$ and by the conjugacy class equation $\kappa - \nu_{i_1} \leq \omega$. Therefore $\frac{a}{b}q^{\omega} < \frac{a}{b}q^{\omega}$, a contradiction.

Proof of (d) Since Γ_F is regular and F is an AC-group, we have $\omega(F) = \frac{q^{\kappa-\omega}-1}{q^{\nu-\omega}-1}$. Therefore $\nu - \omega$ divides $\kappa - \omega$. Now, by considering the conjugacy class equation of F, we find that $\nu - \omega \leq \omega$ and $\kappa \leq 3\omega$.

Now we have two different possibilities on the centralizer orders of H:

(I) $bq^{\nu} > aq^{\omega}$. Since $\Gamma_P \cong \Gamma_H$, we have

$$p^{n-\alpha} - p^{n-\beta} = bq^{\nu} - aq^{\omega},$$

where $\beta = a_t$. It follows from the latter equation, Lemma 2.4 and parts **(b)**,**(d)** that

$$p^{n-\beta-r}|q^{\nu-\omega} - \frac{a}{b} < q^{\omega}|p^u - 1 < p^{n-\beta-r},$$

a contradiction.

(II) $aq^{\omega} > bq^{\nu}$.

Since $\Gamma_P \cong \Gamma_H$, we have

$$aq^{\omega} - bq^{\nu} = p^{n-\beta} - p^{n-\alpha}. \tag{8}$$

We consider two cases:

(i) $u < n - \alpha - r$. Since $u \mid n - \alpha - r$, $2u \leq n - \alpha - r$. Since $H/Z(H)$ is a Frobenius group, $|K/Z(H)| = a/b$ divides $|F/Z(F)| - 1$. Now it follows from parts **(b)** and **(d)**, Lemma 2.4(1) and Equation (8), we have

$$p^{n-\alpha-r} \mid \frac{a}{b} - q^{\nu-\omega} \leq q^{\kappa-\omega} - 1 - q^{\nu-\omega} < q^{2\omega}|(p^u - 1)^2 < p^{2u},$$

a contradiction.

(ii) $u = n - \alpha - r$. Since $u \mid n - \beta - r$, $n - \beta - r \geq 2u$. By the graph isomorphism

$$p^n - p^{n-\beta} = aq^{\omega}(q^{\kappa-\omega} - 1).$$

This latter equation, Lemma 2.4 and parts **(b)** and **(d)** imply that

$$p^{n-\beta-r} \mid q^{\kappa-\omega} - 1 < q^{2\omega} \mid (p^u - 1)^2 < p^{2u},$$

a contradiction.

This completes the proof. □

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